Equivalent Norms

Let a linear space L be formed as a NLS with two norms denoted by $\|x\|_1$ and $\|x\|_2$. Then these two norms are said to be equivalent iff they generate the same topology. We write equivalent norms as—

$$\|x\|_1 \sim \|x\|_2 \text{ or } \|\cdot\|_1 \sim \|\cdot\|_2.$$

Th1: Let N be a NLS and suppose two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are defined on N. Then these norms are equivalent iff $\exists$ +ve real numbers m and M s.t.

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1 \quad , \forall x \in N$$

Pf: Let N_i be NLS with the norm $\|\cdot\|_i$; $i = 1, 2$.

Let T be a L.T. with domain N_1 and range N_2. Definition of T is $T(x) = x$

Then T^{-1} is also linear with N_2 as domain and N_1 its range.

we have→ $\quad T(x) = x$

Then $\quad x = T^{-1}(x)$ or $T^{-1}(x) = x$

So we have→

$$T : N_1 \to N_2 \quad \text{s.t.} \quad T(x) = x$$

$$\& \quad T^{-1} N_2 \to N_1 \quad \text{s.t.} \quad T^{-1}(x) = x$$

Now T is continuous $\Leftrightarrow$ T is bdd.

$\Leftrightarrow \exists$ a +ve number M such that

$\|T(x)\|_2 \leq M \|x\|_1 \quad \forall x \in N_1$

$\Leftrightarrow \|x\|_2 \leq M \|x\|_1 , \quad \forall x \in N_1$

$\because T(x) = x$

$\to (1)$

Again T^{-1} is continuous $\Leftrightarrow T^{-1}$ is bdd.

$\Leftrightarrow \exists$ a +ve number K s.t.

$\|T^{-1}(x)\|_1 \leq K \|x\|_2 \quad \forall x \in N_2$

$\Leftrightarrow \|x\|_1 \leq K \|x\|_2$

$, \because T^{-1}(x) = x$

$\Leftrightarrow \frac{1}{K} \|x\|_1 \leq \|x\|_2$

$\Leftrightarrow m\|x\|_1 \leq \|x\|_2$, letting $\frac{1}{K} = m$ as $K \neq 0$

$\to (2)$

Since T and T^{-1} are continuous

$\Leftrightarrow$ inverse images of open sets in N_2 and N_1 under T and T^{-1} respectively, are open in N_1 and N_2.

$\Leftrightarrow$ open sets in N_1 and N_2 are same since T and T^{-1} are identity transformations

[Note: Both map x onto x]

$\Leftrightarrow \|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on N. $\to (3)$

$\therefore$ From (1), (2), (3) $\to \|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

$\therefore \exists$ +ve numbers m & M st. $m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1 \; \forall x \in N$ [illegible]

Scanned by CamScanner

Note → on a finite dimensional space, all norms are equivalent.

Note → If N is a finite dimensional linear space then N is complete

Note → If N is a NLS and M is any finite dimensional subspace of N then M is closed.

Thm → Let N & N' be NLS and let $T: N \to N'$ be any L.T.
If N is finite dimensional then T is continuous (or bdd.)

Pf → Let $\dim N = n$, since N is finite dimensional

Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for N.
Then we can write any element $x \in N$ as →

$$x = \sum_{i=1}^{n} \alpha_i e_i$$

where α's are unique scalars.

Since T is linear,

$$\therefore \quad T(x) = T\left(\sum_{i=1}^{n} \alpha_i e_i\right)$$

$$= \sum_{i=1}^{n} \alpha_i T(e_i) \quad \to (1)$$

Since all norms are equivalent on a finite dimensional space, ∴ we shall consider one norm say zeroth norm on N defined by —

$\|x\|_0 = \max |\alpha_i|$, $i = 1, 2 \ldots, n$ $\rightarrow (2)$

Now we shall prove that T is bdd under this norm and consequently continuous.

From (1), we get →

$$\|T(x)\| = \|\sum_{i=1}^{n} \alpha_i e_i\|$$

$$\leq \sum_{i=1}^{n} |\alpha_i| \, \|e_i\|$$

$$\leq \|x\|_0 \sum_{i=1}^{n} \|e_i\| \quad \rightarrow (3)$$

Since the basis is fixed,

$\sum_{i=1}^{n} \|e_i\|$ is a +ve const..

Hence, letting $m = \sum_{i=1}^{n} \|e_i\|$,

we get →

$$\|T(x)\| \leq m \|x\|_0$$

$\Rightarrow$ T is bdd.

$\Rightarrow$ T is continuous.

(Proved)

Riesz-Lemma

Let M be a closed proper subspace of a NLS N and let 'a' be a real number such that $0 < a < 1$. Then $\exists$ a vector $x_a \in N$ s.t.

$$\|x_a\| = 1 \quad \text{and} \quad \|x - x_a\| \geq a \quad \forall x \in M.$$

Proof → Let $x_1 \in N - M$ i.e. $x_1 \in N$, $x_1 \notin M$ → (1)

Let $h = \inf \|x - x_1\| = d(x_1, M)$; $x \in M$

h must be > 0 as if $h = 0$

then $d(x_1, M) = 0 \Rightarrow x_1 \in M$, $\because M$ is closed

which is a contradiction

as $x_1 \notin M$, from (1). $\therefore h > 0$

Now $0 < a < 1$

$$\therefore \frac{h}{a} > h$$

$\therefore$ by the defn of infimum, $\exists\, x_0 \in M$ s.t. $h \leq \|x_0 - x_1\| \leq a^{-1} h$ → (2)

Let $x_a = k(x_1 - x_0)$

where $k = \|x_1 - x_0\|^{-1} > 0$

Here $x_1 \neq x_0$, $\because x_1 \in N - M$ and $x_0 \in M$

$\therefore \|x_a\| = k\|x_1 - x_0\| = k k^{-1} = 1$ → (3)

We have proved that $\|x_a\| = 1$

Now we shall prove that

$$\|x - x_a\| \geq a \quad \forall x \in M$$

For this, let $x \in M$ be arbitrary.

Then $k^{-1}x + x_0 \in M$ as M is linear

Now →

$$\|x - x_a\| = \|x - k(x_1 - x_0)\|$$

$$= k\|k^{-1}x - (x_1 - x_0)\|$$

$$= k\|\underbrace{(k^{-1}x + x_0)}_{\in M} - x_1\|$$

$$\therefore \geq kh \quad \rightarrow (4)$$

as $h = \inf \|x - x_1\|$

and $k^{-1}x + x_0 \in M$

so $\|(k^{-1}x + x_0) - x_1\| \geq h$

But $kh = \|x_1 - x_0\|^{-1} h \geq a$, from (2)

$\rightarrow (5)$

$\therefore$ From (4) & (5), we get →

$$\|x - x_a\| \geq a \quad \forall x \in M$$

(Proved)